

The Hopf algebra structure of the \mathbb{Z}_3 -graded quantum supergroup $\mathrm{GL}_{q,j}(1|1)$

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In this work, we give some features of the \mathbb{Z}_3 -graded quantum supergroup.

I. INTRODUCTION

Recently, there have been many attempts to generalize \mathbb{Z}_2 -graded constructions to the \mathbb{Z}_3 -graded case^{1–3}. The \mathbb{Z}_3 -graded quantum space that generalizes the \mathbb{Z}_2 -graded space called a superspace⁴, was studied using the methods of Ref. 5. The first author studied the noncommutative geometry of the \mathbb{Z}_3 -graded quantum superplane⁶. In this work, we have investigated the Hopf algebra structure of the \mathbb{Z}_3 -graded quantum supergroup $\mathrm{GL}_{q,j}(1|1)$.

Let us shortly investigate a general \mathbb{Z}_3 -graded algebraic structure. Let z be a \mathbb{Z}_3 -graded variable. Then we say that the variable z satisfies the relation

$$z^3 = 0.$$

If $f(z)$ is an arbitrary function of the variable z , then the function $f(z)$ becomes a polynomial of degree two in z , that is,

$$f(z) = a_0 + a_1 z + a_2 z^2,$$

where a_0, a_2, a_1 denote three fixed numbers whose grades are $\mathrm{grad}(a_0) = 0$, $\mathrm{grad}(a_2) = 1$ and $\mathrm{grad}(a_1) = 2$, respectively.

The cyclic group \mathbb{Z}_3 can be represented in the complex plane by means of the cubic roots of 1: let $j = e^{\frac{2\pi i}{3}}$ ($i^2 = -1$). Then one has

$$j^3 = 1 \quad \text{and} \quad j^2 + j + 1 = 0.$$

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One can define the Z_3 -graded commutator $[A, B]$ as

$$[A, B]_{Z_3} = AB - j^{ab}BA,$$

where $\text{grad}(A) = a$ and $\text{grad}(B) = b$. If A and B are j -commutative, then we have

$$AB = j^{ab}BA.$$

II. REVIEW OF THE ALGEBRA OF FUNCTIONS ON THE Z_3 -GRADED QUANTUM SUPERPLANE

The Z_3 -graded quantum superplane is defined as an associative unital algebra generated by x and θ satisfying⁶

$$x\theta = q\theta x, \quad \theta^3 = 0 \tag{1}$$

where q is a nonzero complex deformation parameter. Here, the coordinate x with respect to the Z_3 -grading is of grade 0 and the coordinate θ with respect to the Z_3 -grading is of grade 1. This associative algebra over the complex numbers is known as the algebra of polynomials over the quantum superplane and we shall denote it by $\mathcal{R}_q(1|1)$, that is,

$$\mathcal{R}_q(1|1) \ni \begin{pmatrix} x \\ \theta \end{pmatrix} \iff x\theta = q\theta x, \quad \theta^3 = 0.$$

If we denote the dual of the $\mathcal{R}_q(1|1)$ by $\mathcal{R}_{q,j}^*(1|1)$, one has

$$\mathcal{R}_{q,j}^*(1|1) \ni \begin{pmatrix} \varphi \\ y \end{pmatrix} \iff \varphi y = qjy\varphi, \quad \varphi^3 = 0. \tag{2}$$

Here,

$$[\mathcal{R}_q(1|1)]^* = \mathcal{R}_{q,j}^*(1|1).$$

We define the extended quantum superplane to be the algebra that contains $\mathcal{R}_q(1|1)$, the unit and x^{-1} , the inverse of x , which obeys

$$xx^{-1} = 1 = x^{-1}x.$$

We denote the extended algebra by \mathcal{A} . We know that the algebra \mathcal{A} is a Z_3 -graded Hopf algebra.⁶

III. A PERSPECTIVE TO Z_3 -GRADED h -DEFORMATION

We know that the commutation relation between the coordinate x' and the coordinate θ' of the Z_3 -graded quantum superplane is in the form

$$x'\theta' - q\theta'x' = 0.$$

We now introduce new coordinates x and θ , in terms of x' and θ' as

$$x = x', \quad \theta = \theta' - \frac{h}{q-1}x' \quad (3)$$

as in Ref. 7. This transformation is singular in the $q \rightarrow 1$ limit. Using relation (1), it is easy to verify that

$$x\theta = q\theta x + hx^2 \quad (4)$$

where the new deformation parameter h commutes with the coordinate x . Also, since the grassmann coordinate θ' satisfies

$$\theta'^3 = 0$$

one obtains

$$\theta^3 = 0 \quad (5)$$

provided that

$$\theta h = qjh\theta, \quad h^3 = 0. \quad (6)$$

Taking the $q \rightarrow 1$ limit we obtain the following relations which define the Z_3 -graded h -superplane

$$x\theta = \theta x + hx^2, \quad \theta^3 = 0. \quad (7)$$

Also, it can be obtained the Z_3 -graded h -supergroup with Aghamohammadi's approach in Ref. 7. So, it can be investigated the differential geometry of this group. This work is in progress.

IV. Z_3 -GRADED QUANTUM SUPERGROUPS

A. Quantum matrices in Z_3 -graded superspace

In this section, we shall consider the Z_3 -graded structures of the quantum 2x2 supermatrices. We know, from section 2, that the Z_3 -graded quantum superplane $\mathcal{R}_q(1|1)$ is generated by coordinates x and θ , with the commutation rules (1), and the dual Z_3 -graded quantum superplane $\mathcal{R}_{q,j}^*(1|1)$ as generated by φ and y with the relations (2).

Let T be a 2x2 (super)matrix in Z_3 -graded space,

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \quad (8)$$

where a and d with respect to the Z_3 -grading are of grade 0, and β and γ with respect to the Z_3 -grading are of grade 2 and of grade 1, respectively. We now consider linear transformations with the following properties:

$$T : \mathcal{R}_q(1|1) \longrightarrow \mathcal{R}_q(1|1), \quad T : \mathcal{R}_{q,j}^*(1|1) \longrightarrow \mathcal{R}_{q,j}^*(1|1). \quad (9)$$

We assume that the entries of T are j -commutative with the elements of $\mathcal{R}_q(1|1)$ and $\mathcal{R}_{q,j}^*(1|1)$, i.e. for example,

$$ax = xa, \quad \theta\beta = j^2\beta\theta,$$

etc. As a consequence of the linear transformations in (9) the elements

$$\tilde{x} = ax + \beta\theta, \quad \tilde{\theta} = \gamma x + d\theta \quad (10)$$

should satisfy the relations (1):

$$\tilde{x}\tilde{\theta} = q\tilde{\theta}\tilde{x}, \quad \tilde{\theta}^3 = 0.$$

Using these relations , one has

$$\begin{aligned} a\gamma &= q\gamma a, & d\gamma &= q\gamma d, \\ d\beta &= q^{-1}j\beta d, & \gamma^3 &= 0. \end{aligned}$$

Similarly, the elements

$$\tilde{\varphi} = a\varphi + j^2\beta y, \quad \tilde{y} = j\gamma\varphi + dy \quad (11)$$

must be satisfy the relations (2). Using these relations , one has

$$a\beta = q^{-1}j^{-1}\beta a, \quad \beta^3 = 0.$$

Also, if we use the following relation in Ref. 6 (see, page 4262, eq. (19))

$$\tilde{x}\tilde{y} = q\tilde{y}\tilde{x} + (j^2 - 1)\tilde{\varphi}\tilde{\theta},$$

we have

$$ad = da + q^{-1}(1 - j)\beta\gamma, \quad \beta\gamma = q^2\gamma\beta.$$

Consequently, we have the following commutation relations between the matrix elements of T which is given in Ref. 6:

$$\begin{aligned} a\beta &= q^{-1}j^{-1}\beta a, & d\beta &= q^{-1}j\beta d, \\ a\gamma &= q\gamma a, & d\gamma &= q\gamma d, \\ ad &= da + q^{-1}(1 - j)\beta\gamma, & \beta\gamma &= q^2\gamma\beta, \end{aligned}$$

$$\beta^3 = 0, \quad \gamma^3 = 0. \quad (12)$$

The \mathbb{Z}_3 -graded quantum (super)determinant is defined by

$$D_{q,j}(T) = ad^{-1} + ad^{-1}\gamma a^{-1}\beta d^{-1} + ad^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1} \quad (13)$$

provided d and a are invertible. The commutation relations between the matrix elements of T and its (super)determinant:

$$\begin{aligned} aD_{q,j}(T) &= D_{q,j}(T)a, & \beta D_{q,j}(T) &= j^2 D_{q,j}(T)\beta \\ \gamma D_{q,j}(T) &= D_{q,j}(T)\gamma, & daD_{q,j}(T) &= D_{q,j}(T)ad \end{aligned} \quad (14)$$

If we take

$$\Delta_1 = ad - q^{-1}\beta\gamma \quad \Delta_2 = da - qj\gamma\beta$$

the \mathbb{Z}_3 -graded quantum (super)inverse of T becomes

$$T^{-1} = \begin{pmatrix} d\Delta_1^{-1} & -qj\beta\Delta_2^{-1} \\ -q^{-1}\gamma\Delta_1^{-1} & a\Delta_2^{-1} \end{pmatrix}.$$

Using the relations (12), one can check that the following relations:

$$\begin{aligned} \Delta_1 a &= a\Delta_1, & \Delta_1 d &= d\Delta_1 \\ \Delta_2 a &= a\Delta_2, & \Delta_2 d &= d\Delta_2 \\ \Delta_k \beta &= q^{-2}\beta\Delta_k, & \Delta_k \gamma &= q^2\gamma\Delta_k, & k &= 1, 2 \\ \Delta_1 \Delta_2 &= \Delta_2 \Delta_1. \end{aligned} \quad (15)$$

The \mathbb{Z}_3 -graded quantum (super)determinant of T , according to these facts, is given by

$$D_{q,j}(T) = a^2 \Delta_2^{-1}.$$

Of course, the \mathbb{Z}_3 -graded quantum (super)determinant of T^{-1} may also be defined and it is of the form

$$D_{q,j}(T^{-1}) = d^2 \Delta_1^{-1}.$$

Explicitly,

$$D_{q,j}(T^{-1}) = da^{-1} + da^{-1}\beta d^{-1}\gamma a^{-1} + da^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}. \quad (16)$$

Let's now consider the multiplication of two \mathbb{Z}_3 -graded quantum (super) matrices. If we take them as

$$T_1 = \begin{pmatrix} a_1 & \beta_1 \\ \gamma_1 & d_1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} a_2 & \beta_2 \\ \gamma_2 & d_2 \end{pmatrix}$$

where the matrix elements of T_1 and T_2 satisfy the relations (12). Then the matrix elements of $T_1 T_2$ leave invariant the relations (12), as expected. Here, we assume that the commutation relations between the elements of T_1 and T_2 are as follows

$$\begin{aligned}\beta_1 \gamma_2 &= j \gamma_2 \beta_1, & \beta_1 \beta_2 &= j^2 \beta_2 \beta_1, \\ \gamma_1 \beta_2 &= j \beta_2 \gamma_1, & \gamma_1 \gamma_2 &= j^2 \gamma_2 \gamma_1\end{aligned}$$

and the elements whose gradings are 0 commute with all the other elements. Also, the Z_3 -graded quantum (super)determinant is not central of the Z_3 -graded quantum (super)group, although the Z_2 -graded quantum superdeterminant is central of the Z_2 -graded quantum supergroup with two parameters.⁸

We shall denote with $GL_{q,j}(1|1)$ the quantum supergroup in Z_3 -graded space determined by generators a, β, γ, d satisfying the commutation relations (12).

Also, we can define the Z_3 -graded quantum (super)transpoze of T as

$$T^{st} = \begin{pmatrix} a & \gamma \\ j\beta & d \end{pmatrix}. \quad (17)$$

Note that, the transformations

$$T^{st} : \mathcal{R}_q(1|1) \longrightarrow \mathcal{R}_q(1|1), \quad T^{st} : \mathcal{R}_{q,j}^*(1|1) \longrightarrow \mathcal{R}_{q,j}^*(1|1) \quad (18)$$

with together the transformations (9), will give the relations (12). Here, the action of T^{st} on the coordinate functions as follows

$$\hat{x} = xa + j\theta\beta, \quad \hat{\theta} = x\gamma + \theta d.$$

One can show that $T^{-1} \in GL_{q^{-1},j^{-1}}(1|1)$. Indeed, if we denote the Z_3 -graded quantum (super)inverse of T with

$$T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (19)$$

then we have

$$\begin{aligned}A &= a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}, \\ B &= -a^{-1}\beta d^{-1} - a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1} - a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}, \\ C &= -d^{-1}\gamma a^{-1} - d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}, \\ D &= d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}.\end{aligned} \quad (20)$$

Now, using the relations (15) and (12) one has $T^{-1} \in GL_{q^{-1},j^{-1}}(1|1)$.

The relations between the matrix elements of T and (super)inverse of T are important, because we will use them to set up the Z_3 -graded differential

geometric structure of the Z_3 -graded quantum supergroup.⁹ These relations are as follows:

$$\begin{aligned}
aA &= j^2 Aa + 1 - j^2, & aB &= q^{-1} j^2 Ba, \\
aC &= qCa, & aD &= Da, \\
\beta A &= q^{-1} j^2 A\beta, & \beta B &= q^{-2} B\beta, \\
\beta C &= C\beta, & \beta D &= q^{-1} j D\beta, \\
\gamma A &= qA\gamma, & \gamma B &= B\gamma, \\
\gamma C &= q^2 C\gamma, & \gamma D &= qD\gamma, \\
dA &= Ad, & dB &= q^{-1} j Bd, \\
dC &= qCd, & dD &= jDd + 1 - j.
\end{aligned} \tag{21}$$

Also, it can be investigated some properties of the quantum (super)matrices in the quantum supergroup $GL_{q,j}(1|1)$. So, perhaps any element of $GL_{q,j}(1|1)$ can be expressed as the exponential of a matrix of non-commuting elements, like the group $GL_q(1|1)$. This work is also in progress.

B. Hopf algebra structure of the Z_3 -graded $GL_{q,j}(1|1)$

In this section, we shall build up the Hopf algebra structure of the Z_3 -graded quantum supergroup $GL_{q,j}(1|1)$. For this, we shall introduce three operators Δ , ϵ and S on the $GL_{q,j}(1|1)$, which are called the coproduct (comultiplication), the counit and the coinverse (antipode), respectively. The coproduct

$$\Delta : GL_{q,j}(1|1) \longrightarrow GL_{q,j}(1|1) \otimes GL_{q,j}(1|1)$$

is defined by

$$\Delta(T) = T \dot{\otimes} T \tag{22}$$

where $\dot{\otimes}$ stands for the usual tensor product and the dot refers to the summation over repeated indices and reminds us about the usual matrix multiplication. The coproduct Δ is an algebra homomorphism which is co-associative, that is

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \tag{23}$$

where \circ stands for the composition of maps and id denotes the identity mapping. Also, the multiplication in the algebra of matrix entries of T is defined as

$$(A \otimes B)(C \otimes D) = j^{\text{grad}(B)\text{grad}(C)} AC \otimes BD.$$

The action on the generators of the $GL_{q,j}(1|1)$ of Δ is

$$\Delta(a) = a \otimes a + \beta \otimes \gamma, \quad \Delta(\beta) = a \otimes \beta + \beta \otimes d,$$

$$\Delta(\gamma) = \gamma \otimes a + d \otimes \gamma, \quad \Delta(d) = \gamma \otimes \beta + d \otimes d \quad (24)$$

where \otimes denotes the tensor product. Of course, the coproduct Δ leaves invariant the relations (12). The counit

$$\epsilon : GL_{q,j}(1|1) \longrightarrow \mathcal{C}$$

is defined by

$$\epsilon(T) = I. \quad (25)$$

The action on the generators of $GL_{q,j}(1|1)$ of ϵ is

$$\epsilon(a) = 1, \quad \epsilon(\beta) = 0, \quad \epsilon(\gamma) = 0, \quad \epsilon(d) = 1. \quad (26)$$

The counit ϵ is an algebra homomorphism such that

$$\mu \circ (\epsilon \otimes \text{id}) \circ \Delta = \mu' \circ (\text{id} \otimes \epsilon) \circ \Delta \quad (27)$$

where

$$\mu : \mathcal{C} \otimes GL_{q,j}(1|1) \longrightarrow GL_{q,j}(1|1), \quad \mu' : GL_{q,j}(1|1) \otimes \mathcal{C} \longrightarrow GL_{q,j}(1|1)$$

are the canonical isomorphisms, defined by

$$\mu(c \otimes a) = ca = \mu'(a \otimes c), \quad \forall a \in GL_{q,j}(1|1), \quad \forall c \in \mathcal{C}.$$

Thus, we have verified that $GL_{q,j}(1|1)$ is a bialgebra with the multiplication m satisfying the associativity axiom

$$m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)$$

where $m(a \otimes b) = ab$.

A bialgebra with the extra structure of the coinverse is called a Hopf algebra.¹⁰

The coinverse

$$S : GL_{q,j}(1|1) \longrightarrow GL_{q,j}(1|1)$$

is defined by

$$S(T) = T^{-1}. \quad (28)$$

The coinverse S is an algebra anti-homomorphism which satisfies

$$m \circ (S \otimes \text{id}) \circ \Delta = \epsilon = m \circ (\text{id} \otimes S) \circ \Delta. \quad (29)$$

The coproduct, counit and coinverse which are specified above supply $GL_{q,j}(1|1)$ with a Hopf algebra structure. It can be show that $GL_{q,j}(1|1)$ has a \mathbb{Z}_3 -graded differential geometric structure.⁹

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